Brownian motion and Stochastic Calculus
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Assignment 8—solutions

Exercise 1

Let a and d be positive real numbers and B a standard Brownian motion.

1) Compute for $\lambda > 0$
$$\mathbb{E}^{\mathbb{P}}\Big[\exp\big(-|B_d|\sqrt{2\lambda}\big)\mathbf{1}_{\{B_d\leq -a\}}\Big].$$

2) Define $T_1 := \inf\{t \geq d : B_t = 0\}$. Show that T_1 is an $\mathbb{F}^{B,\mathbb{P}}$–stopping time, and compute for any $\lambda > 0$
$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda T_1)\big],\ \text{and}\ \mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda T_1)\mathbf{1}_{\{B_d\leq -a\}}\big].$$
Show then that B_{T_1+d} is independent of B_d and T_1.

3) We now define τ_1 by
$$\tau_1 := \begin{cases} d,\ \text{if}\ B_d \leq -a,\\ T_1+d,\ \text{if}\ B_d > -a,\ \text{and}\ B_{T_1+d}\leq -a,\\ +\infty,\ \text{otherwise.}\end{cases}$$
Compute for any $\lambda > 0$
$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda\tau_1)\big].$$

4) Let now
$$T_2 := \inf\{t \geq T_1+d : B_t = 0\}.$$
As above we then introduce
$$\tau_2 := \begin{cases} d,\ \text{if}\ B_d \leq -a,\\ T_1+d,\ \text{if}\ B_d > -a,\ \text{and}\ B_{T_1+d}\leq -a,\\ T_2+d,\ \text{if}\ B_d > -a,\ B_{T_1+d} > -a,\ \text{and}\ B_{T_2+d}\leq -a,\\ +\infty,\ \text{otherwise.}\end{cases}$$
Show that B_{T_2+d} is independent of (B_{T_1+d}, B_d) and T_2, then compute for any $\lambda > 0$
$$\mathbb{E}\big[\exp(-\lambda\tau_2)\big].$$

1) **We directly have**
$$\mathbb{E}^{\mathbb{P}}\Big[\exp\big(-|B_d|\sqrt{2\lambda}\big)\mathbf{1}_{\{B_d\leq -a\}}\Big] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-a/\sqrt{d}} \mathrm{e}^{-\sqrt{2\lambda d}|x|}\mathrm{e}^{-x^2/2}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{a/\sqrt{d}}^{\infty}\mathrm{e}^{-(x+\sqrt{2\lambda d})^2/2+\lambda d}\mathrm{d}x$$
$$= \big(1-\phi\big(a/\sqrt{d}+\sqrt{2\lambda d}\big)\big)\mathrm{e}^{\lambda d}.$$

2) **The fact that T_1 is stopping time is standard. We also have $T_1 \overset{\text{law}}{=} d+\inf\{t\geq 0 : B_{t+d}-B_d = -B_d\}$. By the weak Markov property and the time-invariance of Brownian motion, we know that $W_\cdot := B_{\cdot+d}-B_d$ is a Brownian motion independent of B_d. Now, conditionally on B_d, the law of T_1-d becomes the law of the first hitting time of $-B_d$ by the Brownian motion W. Again by symmetry, this is the same as the first hitting time of $|B_d|$, for which we know the Laplace transform by Assignment 6. Namely for any $\lambda\geq 0$**
$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda(T_1-d))\big|B_d\big] = \exp\big(-|B_d|\sqrt{2\lambda}\big).$$

Thus we have that

$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda T_1)\big] = \mathbb{E}^{\mathbb{P}}\Big[\exp\big(-|B_d|\sqrt{2\lambda}\big)\Big]\mathrm{e}^{-\lambda d} = \frac{\mathrm{e}^{-\lambda d}}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{-\sqrt{2\lambda d}|x|}\mathrm{e}^{-x^2/2}\mathrm{d}x$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{e}^{-(x+\sqrt{2\lambda d})^2/2}\mathrm{d}x$$
$$= \sqrt{\frac{2}{\pi}}\int_{\sqrt{2\lambda d}}^\infty \mathrm{e}^{-x^2/2}\mathrm{d}x$$
$$= 2\big(1-\phi\big(\sqrt{2\lambda d}\big)\big).$$

Similarly, using now 1)

$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda T_1)\mathbf{1}_{\{B_d\leq -a\}}\big] = \mathrm{e}^{-\lambda d}\mathbb{E}^{\mathbb{P}}\Big[\exp\big(-|B_d|\sqrt{2\lambda}\big)\big)\mathbf{1}_{\{B_d\leq -a\}}\Big] = 1-\phi\big(a/\sqrt{d}+\sqrt{2\lambda d}\big).$$

Now notice that by the strong Markov property, $B_{T_1+d}-B_{T_1} = B_{T_1+d}$ **is independent of** $\mathcal{F}_{T_1}$**, and thus in particular of** B_d **and** T_1 **(since** T_1 **is** $\mathcal{F}_{T_1}$**-measurable).**

3) **We have, using that** B_{T_1+d} **is independent of** B_d **and** T_1

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda\tau_1)\big] &= \mathrm{e}^{-\lambda d}\mathbb{P}[B_d\leq -a] + \mathrm{e}^{-\lambda d}\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{-\lambda T_1}\mathbf{1}_{\{B_d>-a\}\cap\{B_{T_1+d}\leq -a\}}\big]\\ &= \mathrm{e}^{-\lambda d}\phi\big(-a/\sqrt{d}\big) + \mathrm{e}^{-\lambda d}\mathbb{P}[B_{T_1+d}\leq -a]\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{-\lambda T_1}\mathbf{1}_{\{B_d>-a\}}\big]\\ &= \mathrm{e}^{-\lambda d}\phi\big(-a/\sqrt{d}\big) + \mathrm{e}^{-\lambda d}\big(1-2\phi\big(\sqrt{2\lambda d}\big)+\phi\big(a/\sqrt{d}+\sqrt{2\lambda d}\big)\big)\mathbb{E}^{\mathbb{P}}\big[\phi\big(-a/\sqrt{T_1+d}\big)\big].\end{aligned}$$

4) **We argue as in** 3) **to get that** $B_{T_2+d}-B_{T_2} = B_{T_2+d}$ **is independent of** $\mathcal{F}_{T_2}$**, and thus of** B_d**,** B_{T_1+d} **and** T_2**. We also have that** T_2 **and** B_d **themselves are independent. Then**

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda\tau_2)\big] &= \mathrm{e}^{-\lambda d}\mathbb{P}[B_d\leq -a] + \mathrm{e}^{-\lambda d}\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{-\lambda T_1}\mathbf{1}_{\{B_d>-a\}\cap\{B_{T_1+d}\leq -a\}}\big]\\ &\quad + \mathrm{e}^{-\lambda d}\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{-\lambda T_2}\mathbf{1}_{\{B_d>-a\}\cap\{B_{T_1+d}>-a\}\cap\{B_{T_2+d}\leq -a\}}\big]\\ &= \mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda\tau_1)\big] + \mathrm{e}^{-\lambda d}\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{-\lambda T_2}\mathbf{1}_{\{B_d>-a\}\cap\{B_{T_1+d}>-a\}}\big]\mathbb{P}[B_{T_2+d}\leq -a]\\ &= \mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda\tau_1)\big] + \mathrm{e}^{-\lambda d}\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{-\lambda T_2}\mathbf{1}_{\{B_{T_1+d}>-a\}}\big]\big(1-\phi\big(-a\sqrt{d}\big)\big)\mathbb{E}^{\mathbb{P}}\big[\phi\big(-a/\sqrt{T_2+d}\big)\big]\end{aligned}$$

Notice also that as in 3)**, we have** $T_2 \overset{\text{law}}{=} T_1+d+\inf\big\{t\geq 0: B_{t+T_1+d}-B_{T_1+d} = -B_{T_1+d}\big\}$**. Thus**

$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\lambda T_2)\big|B_{T_1+d},T_1\big] = \mathrm{e}^{-\lambda(T_1+d)}\exp\big(-|B_{T_1+d}|\sqrt{2\lambda}\big),$$

so that using the independence between T_1 **and** B_{T_1+d}

$$\mathbb{E}^{\mathbb{P}}[\exp(-\lambda T_2)] = \mathrm{e}^{-\lambda d}\mathbb{E}^{\mathbb{P}}[\exp(-\lambda T_1)]\mathbb{E}^{\mathbb{P}}\Big[\exp\big(-|B_{T_1+d}|\sqrt{2\lambda}\big)\Big] = 4\mathrm{e}^{-\lambda d}\big(1-\phi\big(\sqrt{2\lambda d}\big)\big)\Big(1-\mathbb{E}^{\mathbb{P}}\big[\phi\big(\sqrt{2\lambda(T_1+d)}\big)\big]\Big).$$

Exercise 2

Let B be a standard one-dimensional Brownian motion $(B_t)_{t\geq 0}$. We define

$$X_t := \frac{1}{t}\int_0^t \mathbf{1}_{\{B_s>0\}}\mathrm{d}s,\ t>0.$$

Our goal is to show that

$$\mathbb{P}[X_t<u] = \frac{2}{\pi}\mathrm{Arcsin}(\sqrt{u}),\ 0\leq u\leq 1,\ t>0.$$

1) What does X_t represent?

2) Show that the law of X_t is equal to the law of X_1, for any $t > 0$.

3) We fix $\lambda > 0$ and define for $(t, x) \in]0, +\infty[\times\mathbb{R}$ the map

$$v(t,x) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(- \lambda \int_0^t \mathbf{1}_{\{x+B_s>0\}}\mathrm{d}s\right)\right],$$

as well as its Laplace transform

$$g_\rho(x) := \int_0^{+\infty} v(t,x)\mathrm{e}^{-\rho t}\mathrm{d}t,\ \rho > 0.$$

Show that

$$g_\rho(0) = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\rho + \lambda X_1}\right].$$

4) Assuming that all functions appearing are smooth enough, show that v must satisfy

$$\frac{\partial v}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x) - \lambda \mathbf{1}_{\{x>0\}}v(t,x).$$

5) Deduce then that g_ρ must satisfy

$$g''_\rho(x) = -2 + 2\rho g_\rho(x) + 2\lambda \mathbf{1}_{x>0} g_\rho(x).$$

6) Solve this ODE on $\mathbb{R}$, and deduce in particular that

$$g_\rho(0) = \frac{1}{\sqrt{\rho(\lambda+\rho)}}.$$

7) Deduce that the result stated at the beginning of the exercise holds. You may want to use (and prove!) the following identity

$$\frac{1}{\sqrt{1+\lambda}} = \frac{1}{\pi}\sum_{n=0}^{+\infty}(-\lambda)^n \int_0^1 \frac{x^n}{\sqrt{x(1-x)}}\mathrm{d}x.$$

1) **This is the average time that B spends above 0.**

2) **By the scaling invariance of B, we have**

$$X_t = \int_0^1 \mathbf{1}_{\{B_{tu}>0\}}\mathrm{d}u \overset{\text{law}}{=} \int_0^1 \mathbf{1}_{\{\sqrt{t}B_u>0\}}\mathrm{d}u = \int_0^1 \mathbf{1}_{\{B_u>0\}}\mathrm{d}u = X_1.$$

3) **We have using Fubini's theorem and $1)$**

$$\begin{aligned} g_\rho(0) &= \int_0^{+\infty} \mathbb{E}^{\mathbb{P}}\left[\exp\left(- \lambda \int_0^t \mathbf{1}_{\{B_s>0\}}\mathrm{d}s\right)\right]\mathrm{e}^{-\rho t}\mathrm{d}t \\ &= \mathbb{E}^{\mathbb{P}}\left[\int_0^{+\infty} \exp(-\lambda t X_1)\mathrm{e}^{-\rho t}\mathrm{d}t\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\frac{1}{\rho+\lambda X_1}\right]. \end{aligned}$$

4) **Here one has to recognise that this is an application of Feynman–Kac formula (with a time-reversal to have a boundary condition at $t = 0$ instead of $t - T$. More precisely, fix some $\tilde{v}$ solving the PDE with the boundary condition $\tilde{v}(0,\cdot) = v(0,\cdot) = 1$. Let us then define for some given $T > 0$**

$$u(t,x) := \tilde{v}(T-t,x).$$

It is immediate that u solves

$$\frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) - \lambda \mathbf{1}_{\{x>0\}} u(t,x) = 0,\ (t,x) \in [0,T) \times \mathbb{R},\ u(T,x) = 1,\ x \in \mathbb{R}.$$

Then, assuming smoothness, Feynman–Kac's formula tells us that

$$u(t,x) = \mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(-\lambda \int_t^T \mathbf{1}_{\{x+B_{s-t}>0\}}\mathrm{d}u\bigg)\bigg],$$

so that

$$\tilde{v}(t,x) = u(T-t,x) = \mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(-\lambda \int_{T-t}^T \mathbf{1}_{\{x+B_{s-T+t}>0\}}\mathrm{d}u\bigg)\bigg] = v(t,x).$$

5) **We differentiate formally and then integrate by parts, recalling that v is bounded**

$$\begin{aligned} g_\rho''(x) = \int_0^{+\infty} \frac{\partial^2 v}{\partial x^2}(t,x)\mathrm{e}^{-\rho t}\mathrm{d}t &= \int_0^{+\infty} \bigg(2\frac{\partial v}{\partial t}(t,x) + \lambda \mathbf{1}_{\{x>0\}} v(t,x)\bigg)\mathrm{e}^{-\rho t}\mathrm{d}t \\ &= 2\big[v(\cdot,x)\mathrm{e}^{-\rho \cdot}\big]_0^{+\infty} + 2\rho\int_0^{+\infty} \mathrm{e}^{-\rho t} v(t,x)\mathrm{d}t + 2\lambda \mathbf{1}_{x>0} g_\rho(x) \\ &= -2 + 2\rho g_\rho(x) + 2\lambda \mathbf{1}_{x>0} g_\rho(x). \end{aligned}$$

6) **We will solve the ODE on $(-\infty,0)$ and $(0,+\infty)$ separately and try to paste the solutions together. First, the general solution to the (linear) ODE on $(-\infty,0)$ is classically given by (recall ρ is positive)**

$$\bar{g}_\rho(x) = A\exp\big(\sqrt{2\rho}x\big) + B\exp\big(-\sqrt{2\rho}x\big) + \frac{1}{\rho},\ \textbf{for arbitrary } (A,B) \in \mathbb{R}^2\textbf{.}$$

Similarly, the general solution on $(0,+\infty)$ is

$$\hat{g}_\rho(x) = C\exp\big(\sqrt{2(\rho+\lambda)}x\big) + D\exp\big(-\sqrt{2(\rho+\lambda)}x\big) + \frac{1}{\rho+\lambda},\ \textbf{for arbitrary } (C,D) \in \mathbb{R}^2\textbf{.}$$

Now recall that since v is bounded, some must be g_ρ, which implies that $B = C = 0$. Now we have

$$\hat{g}_\rho(0) = \bar{g}_\rho(0) \Longleftrightarrow A + \frac{1}{\rho} = D + \frac{1}{\rho+\lambda} \Longleftrightarrow D = \frac{\lambda}{\rho(\rho+\lambda)} + A.$$

Now if we also want the solution to be differentiable at 0, we must have

$$A\sqrt{2\rho} = -\bigg(\frac{\lambda}{\rho(\rho+\lambda)} + A\bigg)\sqrt{2(\rho+\lambda)} \Longleftrightarrow A = -\frac{\sqrt{2(\rho+\lambda)} - \sqrt{2\rho}}{\rho\sqrt{2(\rho+\lambda)}} = -\frac{1}{\rho} + \frac{1}{\sqrt{\rho(\lambda+\rho)}}.$$

This completely characterises a C^1 solution of the ODE, which is C^2 on $\mathbb{R}^\star$. In particular, we have as desired

$$g_\rho(0) = \frac{1}{\sqrt{\rho(\lambda+\rho)}}.$$

7) **We first show that for any $n \in \mathbb{N}$**

$$\int_0^1 \frac{x^n}{\sqrt{x(1-x)}}\mathrm{d}x = \pi\frac{(2n)!}{4^n(n!)^2},$$

and the stated equality then stems from the known Taylor series for the inverse square root

$$(1+\lambda)^{-1/2} = \sum_{n=0}^{+\infty}(-1)^n\frac{(2n)!}{4^n(n!)^2}\lambda^n.$$

As for the first claim, one simply needs to recognise that

$$\int_0^1 \frac{x^n}{\sqrt{x(1-x)}}\mathrm{d}x = B\left(n+\frac{1}{2},\frac{1}{2}\right),$$

where B is the Beta function, which can be more simply rewritten in terms of Euler's Gamma function

$$B\left(n+\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(n+1/2)\Gamma(1/2)}{\Gamma(n+1)} = \frac{\frac{(2n)!\sqrt{\pi}}{4^n n!}\sqrt{\pi}}{n!} = \pi\frac{(2n)!}{4^n(n!)^2},$$

as desired.

Let now Y be a continuous random variable with

$$\mathbb{P}[Y<u] = \frac{2}{\pi}\mathrm{Arcsin}(\sqrt{u}),\ 0\le u\le 1.$$

One can check directly that Y has a density given by $f(u) := \left(\pi\sqrt{u(1-u)}\right)^{-1}\mathbf{1}_{(0,1)}(u)$. Now what we want to show here is that the $v(t,0)$ is the Laplace transform of Y, that is to say, using the series expansion of the exponential, and standard arguments to invert the summation and the integral

$$v(t,0) = \int_0^1 \mathrm{e}^{-\lambda t u}f(u)\mathrm{d}u = \frac{1}{\pi}\int_0^1\sum_{n=0}^{+\infty}\frac{(-\lambda t)^n}{n!}\frac{u^n}{\sqrt{u(1-u)}}\mathrm{d}u = \sum_{n=0}^{+\infty}\frac{(-\lambda t)^n}{n!}\frac{(2n)!}{4^n(n!)^2}.$$

Now we know that the Laplace transform of $v(\cdot,0)$ is $g_\rho(0)$, so we just need to compute the Laplace transform of the right-hand side above

$$\int_0^{+\infty}\mathrm{e}^{-\rho t}\sum_{n=0}^{+\infty}\frac{(-\lambda t)^n}{n!}\frac{(2n)!}{4^n(n!)^2}\mathrm{d}t = \frac{1}{\rho}\sum_{n=0}^{+\infty}\left(-\frac{\lambda}{\rho}\right)^n\frac{(2n)!}{4^n(n!)^2} = \frac{1}{\rho}\frac{1}{\sqrt{1+\lambda/\rho}} = g_\rho(0),$$

as desired.